



Concepts of Real Analysis

S E C O N D R E V I S E D E D I T I O N

(For Undergraduate and Postgraduate Students of
All Indian Universities and Technical Institutes Under NEP 2020)

Weierstrass Function



Everywhere Continuous, x Nowhere Differentiable

R.D. Sarma
Ankit Gupta • Rajesh Singh

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Concepts of
Real Analysis

Dedicated to

- (i) Dr. Naseem Ajmal, my College Professor and Ph. D. Supervisor
and
- (ii) Mr. Ishita Ranjan Mishra, my High School Mathematics teacher
(the two beacons of my life and career)

— R. D. Sarma



To my Parents
Ms. Sunita Gupta and Mr. Mahavir Prasad Gupta
(To whom I owe everything)

— Ankit Gupta



Dedicated to my family for all their love and support.

— Rajesh Singh

Concepts of Real Analysis

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PREFACE

Preface to the Second Edition

The overwhelming response to the First Edition of *Concepts of Real Analysis* has been both gratifying and humbling to us as authors. It is a privilege to know that our approach to demystifying the foundational rigors of analysis has resonated well with the students and the teachers alike. This second edition is born out of a desire to build upon that foundation, guided by the invaluable feedback, classroom experiences, and insights shared by readers over the past few years.

While the primary objective remains unchanged – to guide students seamlessly from intuitive, calculus-based thinking to the rigorous, proof-oriented mindset required for advanced mathematics – this edition introduces several key improvements and expansions:

- *Expanded Topical Coverage:* Responding to the evolving needs of modern university curricula, particularly with the introduction of National Education Policy 2020, we have significantly enriched the core chapters. In Chapter 11, we have introduced and discussed Weierstrass Approximation Theorem, Exponential and Logarithmic functions as well as Trigonometric functions. Similarly, as an extension of Riemann integration, we have introduced Riemann-Steiltjes integration. This is provided in Appendix C of the book. Applications to the theory of integration is explained through finding the volume of solids of revolution, length of plane curves and area of surfaces of revolutions, thus bridging the gap between classical analysis and its practical, real-world applications.
- *Clarity and Refinement:* Every page has been meticulously reviewed for mathematical precision, clarity of expression, and typographical consistency. Minor ambiguities from the first edition have been ironed out to ensure a smoother, uninterrupted reading experience.

We owe a profound debt of gratitude to our colleagues and peer reviewers whose constructive critiques have directly shaped this revision. Most importantly, we want to thank the students whose curiosity and questions continue to inspire our commitment to mathematical education.

Our publisher M/S Sultan Chand and Sons and specially Mr. Pratap Vaish, Dr. Shubhra and Mr. GD Chaudhury deserve special thanks for their trust on us and the enthusiasm shown by them in bringing out the second edition of the book.

It is our sincere hope that this updated edition continues to serve as an inspiring and reliable companion for the students, teachers and researchers alike in the field of real analysis.

R. D. Sarma
Ankit Gupta
Rajesh Singh

June, 2026

Preface to the First Edition

Real analysis is currently being taught in all Indian universities as a part of the mathematics curricula for various under-graduate programs including B.Sc. (H), B.A. (Prog.), B.Sc (Prog.) etc. A sound knowledge of the limiting concepts, which is the core of real analysis, is a pre-requisite for further study of abstract

mathematical ideas. With time, it has become more pertinent due to the applications of mathematical theories in several important upcoming fields including Computer Science, Bio-Sciences, Genetics, Financial Modelling etc. Our experience shows that students with clear understanding of the basic concepts of analysis find it easier to grasp the abstract ideas later in these fields.

The book covers a wide range of topics including the real numbers, sequences, series, limit, continuity, differentiability, Riemann integrals, improper integrals, sequence and series of functions, power series etc. They are spread over eleven chapters in the book. A good number of examples are provided so that definitions and results can be easily understood by the students. Our experience has shown that students usually find it easier to comprehend through graphs and diagrams. Hence graphs and diagrams are provided at many places in the book. Exercises are provided at the end of each section so that students may have sufficient practice of problem solving. Hints and answers of these exercises are provided towards the end of the book.

Overall, the book is student-friendly, yet full of rigour. We hope that the students as well as the teachers will feel happy to carry this book into the classroom.

The authors take this opportunity to thank the people who helped them during the preparation of the book. We whole-heartedly thank Prof. C. S. Lalitha, Department of Mathematics, University of Delhi for her unfailing guidance and encouragement.

The first author would like to thank Prof. Rajesh Giri, Principal, Rajdhani College, University of Delhi for his good wishes, colleagues Mrs. Krishna Threja, Dr. Umesh Kumar, Prof. Pankaj Kumar Garg, Dr. K. K. Arora, Mr. Biswajit Tahu for their help and co-operation. Special thanks are due to his former students Deepak Tiwari, Piyush Bansal, Rohan Kumar and others for helping him in many ways. His family members, wife Charanjeet and son Jeevant deserve special thanks for their co-operation. He also thanks Mr. Rajeev Arora for providing the word file of the manuscript.

The second author would like to express his heartfelt gratitude to his teachers, his parents, colleagues and students for contributing to his academic journey in different ways. He would like to begin by acknowledging the intellectual and moral support that his research supervisor and also one of the co-authors of this book, Prof. R. D. Sarma, has always provided him. He would like to thank him for numerous invaluable conversations, for constant and critical feedback on this work. He also thinks that he must be failing his duty if he did not express that he remains ever indebted to Dr. Ruchika Verma, Dr. Nidhi Arora, Prof. Iffat Jahan, and Dr. Neelima from Ramjas College (University of Delhi) for not only initiating him into this discipline of mathematics but also for imbibing an interest in it. No journey can be successful without the support and blessings of one's family. The second author would like to thank his parents and family members for their support, faith and inspiration in the hard times. He is also thankful to his friends Mr. Kamal Kumar, Ms. Bhavna Dada, Ms. Chetna Khanna, Ms. Ruchika Gupta, Ms. Neha Mongia, Ms. Nisha Bohra and Ms. Devika for their encouraging words and support which boosted his confidence to achieve this goal.

The third author would like to thank Prof. S. P. Aggarwal, Principal, Ramanujan College for all his encouragement in taking any new project. Special thanks to Prof. R. D. Sarma for believing in him and giving him the opportunity to be part of this project. The author owes an enormous debt of gratitude to his supervisor Prof. Purnima Gupta for inspiring to complete this book and for all her guidance and wishes. He is also immensely grateful to his wife Alka for her constant support and for helping him in proofreading the content. His parents and sisters deserve special thanks for all their co-operation and moral support throughout this endeavour.

Finally, all the authors thank M/s Sultan Chand and Sons and particularly Mr. Pratap Vaish, Mr. G. D. Chaudhary and Dr. Shubhra for bringing out this book with priority in a short span of time.

*R. D. Sarma
Ankit Gupta
Rajesh Singh*

October, 2022
Delhi



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SYMBOLS & GREEK ALPHABETS

$\Rightarrow$	implies
$\Leftrightarrow$	is equivalent to
$\{ \}$	set
$\in$	is an element of
$:$	such that
$\subset$	is contained in (is a subset of)
$\supset$	contains (is a superset of)
$X \sim A$	complement of A with respect to X
$\cup$	union
$\cap$	intersection
ϕ	the empty set
$\exists$	there exists
$\forall$	for all
$\mathbb{N}$	the set of all natural numbers
$\mathbb{Z}$	the set of all integers
$\mathbb{Z}^+$	the set of all positive integers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{Q}^+$	the set of all positive rational numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{R}^+$	the set of all positive real numbers

$\mathbb{C}$	the set of all complex numbers
$\ \cdot \ $	norm
$f^{(n)}$	n^{th} derivative of f
$\chi[a, b]$	characteristic function of $[a, b]$
$\mathcal{P}$	partition of an interval
$U(f, \mathcal{P})$	upper Darboux Sum of f w.r.t $\mathcal{P}$
$L(f, \mathcal{P})$	lower Darboux Sum of f w.r.t $\mathcal{P}$
$S(f, \mathcal{P})$	Riemann sum of f w.r.t $\mathcal{P}$
$\mathcal{R}[a, b]$	the set of all Riemann integrable functions
$\bar{b}$	
$\int_a^{\bar{b}} f$	upper integral of f on $[a, b]$
$\bar{b}$	
$\int_a^b f$	lower integral of f on $[a, b]$
$\underline{a}$	
$\int_a^b f$	integral of f on $[a, b]$
a	

Alpha	A	,	α
Beta	B	,	β
Gamma	Γ	,	γ
Delta	Δ	,	δ
Epsilon	E	,	ε
Zeta	Z	,	ζ
Eta	H	,	η
Theta	Θ	,	θ
Iota	I	,	i
Kappa	K	,	κ
Lambda	Λ	,	λ
Mu	M	,	μ

Nu	N	,	ν
Xi	Ξ	,	ξ
Omicron	O	,	o
Pi	Π	,	π
Rho	P	,	ρ
Sigma	Σ	,	σ
Tau	T	,	τ
Upsilon	Υ	,	υ
Phi	Φ	,	ϕ
Chi	X	,	χ
Psi	Ψ	,	ψ
Omega	Ω	,	ω

About the Book

Concepts of Real Analysis is a student friendly textbook on real analysis; a topic taught as part of undergraduate/postgraduate syllabus of mathematics in all Indian universities and technical institutes. All the major topics of real analysis, such as real numbers, sequence and series of real numbers and of real-valued functions, limit, continuity, applications of derivatives, Riemann integration, power series, etc. are covered in a lucid manner. In addition to this, Riemann-Stieltjes integration, exponential, logarithmic and trigonometric functions are also introduced and discussed in the revised edition. Useful topics from calculus such as double points, asymptotes, curve tracing, length of plane curve, volume and surface area of solids of revolution, etc. are provided as appendices in the book. Each concept is explained with the help of examples. Remarks are provided whenever special attention is required about some aspect of a definition or a result. Diagrams and graphs are provided for further illustration. Overall, it is a complete book in itself, suitable for students, teachers, researchers and aspirants of higher education in mathematics.

Salient Features

- Covers the syllabus of Real Analysis course taught in the Indian universities and technical institutes as per NEP 2020.
- Written in simple language, easy to understand.
- Emphasis on logical, step-by-step explanation of proofs.
- Approximately 500 solved examples and 70 diagrams.
- Sufficient motivations and explanations are provided for the concepts introduced and results proved.
- Remarks are provided to highlight special aspect of a definition or result which might go unnoticed by the readers.
- Student friendly approach.
- Useful concepts from calculus such as curve tracing, area and volume of solids of revolution, length of plane curves, etc. are provided as appendices.

About the Authors

R.D. Sarma is a Professor in the Department of Mathematics, Rajdhani College, University of Delhi. He has more than 30 years of teaching experience in the undergraduate and postgraduate level, in India and abroad. His earlier book "Basic Applied Mathematics for the Physical Sciences" (co-authored) is quite popular among the students of B.Sc. (Prog.). His primary field of research is topology and fuzzy set theory. Professor Sarma has published 50 research papers so far in various international journals. Five students have got Ph.D. degree and two students have got M.Phil. Degree under his supervision.

Ankit Gupta is an Assistant Professor in the Department of Mathematics, Bharati College, University of Delhi. He has a teaching experience of more than 10 years and has taught a variety of subjects from Analysis, Metric Spaces to Mathematical finance at undergraduate level. He received his doctorate degree in Topology from the Department of Mathematics, University of Delhi. He has more than 18 research publications in the area of topology and variational inequality problems in reputed International Journals. He has co-authored another book "Introductory Ring Theory" published by Cambridge University Press.

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