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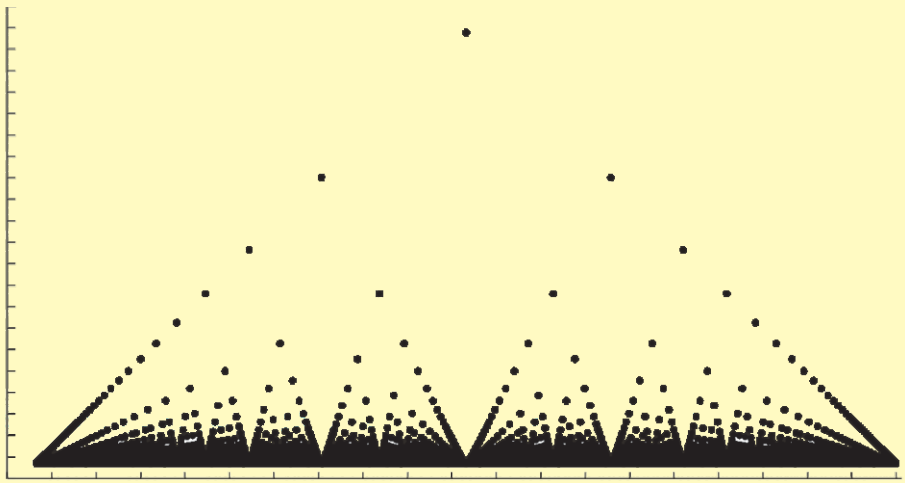
Concepts of
Real Analysis



R. D. Sarma
Ankit Gupta Rajesh Singh

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Concepts of
Real Analysis

Dedicated to

(i) Dr. Naseem Ajmal, my College Professor and Ph. D. Supervisor

and

(ii) Mr. Ishita Ranjan Mishra, my High School Mathematics teacher

(the two beacons of my life and career)

— R. D. Sarma

❑

To my Parents

Ms. Sunita Gupta and Mr. Mahavir Prasad Gupta

(To whom I owe everything)

— Ankit Gupta

❑

Dedicated to my family for all their love and support.

— Rajesh Singh

Concepts of
Real Analysis

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SULTAN CHAND & SONS®
Educational Publishers
New Delhi

SULTAN CHAND & SONS®
23, Daryaganj, New Delhi-110002
Phone: 011-23281876, 23266105, 23277843 (*Showroom & Shop*)
011-40234454, 23247051 (*office*)
E-mail: sultanchand74@yahoo.com; info@sultanchandandsons.com
Fax: 011-23266357; Website: www.sultanchandandsons.com
First Edition: 2023

ISBN: 978-93-91820-27-5 (TC-1266)
Price: ₹ 595.00

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Author's Acknowledgement: The writing of a Textbook always involves creation of a huge debt towards innumerable authors and publications. We owe our gratitude to all of them. We acknowledge our indebtedness in extensive footnotes throughout the book. If, for any reason, any acknowledgement has been left out we beg to be excused. We assure to carry out correction in the subsequent edition, as and when it is known.

Printed at: Himani Print Solution, Badarpur, New Delhi-110044.

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PREFACE

Real analysis is currently being taught in all Indian universities as a part of the mathematics curricula for various under-graduate programs including B.Sc. (H), B.A. (Prog.), B.Sc (Prog.) etc. A sound knowledge of the limiting concepts, which is the core of real analysis, is a pre-requisite for further study of abstract mathematical ideas. With time, it has become more pertinent due to the applications of mathematical theories in several important upcoming fields including Computer Science, Bio-Sciences, Genetics, Financial Modelling etc. Our experience shows that students with clear understanding of the basic concepts of analysis find it easier to grasp the abstract ideas later in these fields.

A good student-friendly book is a basic requirement for developing interest amongst the students for such analytical concepts. Although there are many books written on the subject of real analysis, there are still very few student-friendly books available in the market which can cater to the present syllabus of the Indian universities.

With this background, the authors have come up with the idea of writing a book on real analysis, covering the topics which are presently being taught in the undergraduate level of the Indian universities. The book covers a wide range of topics including the real numbers, sequences, series, limit, continuity, differentiability, Riemann integrals, improper integrals, sequence and series of functions, power series etc. They are spread over eleven chapters in the book. Each chapter is further divided into sections, depending on the variations of the subject matter. Each concept introduced in the book is explained in detail in simple language so that students can grasp it without difficulty. Proofs of the results are provided in an easy, step-by-step manner. A good number of examples are provided so that definitions and results can be easily understood by the students. Our experience has shown that students usually find it easier to comprehend through graphs and diagrams. Hence graphs and diagrams are provided at many places in the book. Students will do well to memorize these graphs. In fact, we have provided appendices to provide the basic concepts of curve tracing in the book. Remarks are used throughout the book to point out and discuss the hidden aspects of the definitions and the theorems, if there is any. Exercises are provided at the end of each section so that students may have sufficient practice of problem solving. Hints and answers of these exercises are provided towards the end of the book.

Overall, the book is student-friendly, yet full of rigour. We hope that the students as well as the teachers will feel happy to carry this book into the classroom.

The authors take this opportunity to thank the people who helped them during the preparation of the book. We whole-heartedly thank Prof. C. S. Lalitha, Department of Mathematics, University of Delhi for her unfailing guidance and encouragement.

The first author would like to thank Prof. Rajesh Giri, Principal, Rajdhani College, University of Delhi for his good wishes, colleagues Mrs. Krishna Threja, Dr. Umesh Kumar, Prof. Pankaj Kumar Garg, Dr. K. K. Arora, Mr. Biswajit Tahu for their help and co-operation. Special thanks are due to his former students Deepak Tiwari, Piyush Bansal, Rohan Kumar and others for helping him in many ways. His family members, wife Charanjeet and son Jeevant deserve special thanks for their co-operation. He also thanks Mr. Rajeev Arora for providing the word file of the manuscript.

The second author would like to express his heartfelt gratitude to his teachers, his parents, colleagues and students for contributing to his academic journey in different ways. He would like to begin by acknowledging the intellectual and moral support that his research supervisor and also one of the co-authors of this book, Prof. R. D. Sarma, has always provided him. He would like to thank him for numerous invaluable conversations, for constant and critical feedback on this work. He also thinks that he must be failing his duty if he did not express that he remains ever indebted to Dr. Ruchika Verma, Dr. Nidhi Arora, Prof. Iffat Jahan, and Dr. Neelima from Ramjas College (University of Delhi) for not only initiating him into this discipline of mathematics but also for imbibing an interest in it. No journey can be successful without the support and blessings of one's family. The second author would like to thank his parents and family members for their support, faith and inspiration in the hard times. He is also thankful to his friends Mr. Kamal Kumar, Ms. Bhavna Dada, Ms. Chetna Khanna, Ms. Ruchika Gupta, Ms. Neha Mongia, Ms. Nisha Bohra and Ms. Devika for their encouraging words and support which boosted his confidence to achieve this goal.

The third author would like to thank Prof. S. P. Aggarwal, Principal, Ramanujan College for all his encouragement in taking any new project. Special thanks to Prof. R. D. Sarma for believing in him and giving him the opportunity to be part of this project. The author owes an enormous debt of gratitude to his supervisor Prof. Purnima Gupta for inspiring to complete this book and for all her guidance and wishes. He is also immensely grateful to his wife Alka for her constant support and for helping him in proofreading the content. His parents and sisters deserve special thanks for all their co-operation and moral support throughout this endeavour.

Finally, all the authors thank M/s Sultan Chand and Sons and particularly Mr. Pratap Vaish, Mr. G. D. Chaudhary and Dr. Shubhra for bringing out this book with priority in a short span of time.

R. D. Sarma
Ankit Gupta
Rajesh Singh

October, 2022
Delhi

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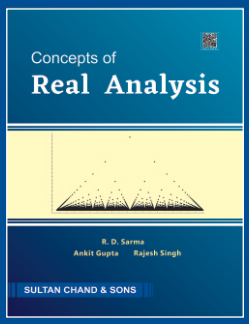
ABOUT THE BOOK



Concepts of Real Analysis is a student friendly textbook on real analysis, a topic taught as part of the undergraduate mathematics syllabus of pass and honours courses of all Indian universities. All the relevant topics of real analysis such as real numbers, sequences and series, limit, continuity, derivatives, Riemann Integration, improper integration, sequence and series of functions, power series etc. are covered in a lucid manner in the book. Each concept is explained with the help of solved examples. Remarks are provided whenever special attention is required about some aspects of a definition or of a result. Diagrams and graphs are provided for further comprehension of a topic or a result, whenever felt necessary. Illustrative examples are provided at the end of each topic, which is followed by exercises. Overall, it is a complete-in-itself book on Real Analysis, suitable for students and teachers alike.

SALIENT FEATURES

- Covers the entire syllabus of Real Analysis taught in the undergraduate level courses including B.Sc. (H.), B.A. (Prog.), B.Sc. (Prog.) of all Indian universities.
- Written in simple language.
- Emphasis on logical, step-by-step development of proofs.
- More than 450 solved examples and 50 diagrams.
- Sufficient explanations are provided for the concepts introduced and results provided.
- Remarks are provided to highlight any special aspect of a definition or a result, which might go unnoticed by the readers.
- Student friendly approach.
- Appendix is added to provide the basics for curve tracing.

ABOUT THE AUTHORS

R. D. Sarma is a Professor in the Department of Mathematics, Rajdhani College, University of Delhi. He has more than 27 years of teaching experience in the undergraduate and postgraduate level, in India and abroad. His earlier book "Basic Applied Mathematics for the Physical Sciences" (co-authored) is quite popular among the students of B.Sc. (Prog.). His primary field of research is topology and fuzzy set theory. Professor Sarma has published 45 research papers so far in various international journals. Three students have got Ph.D. degree and two students have got M.Phil. Degree under his supervision.

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ISBN 978-93-91820-27-5
9 789391 820275