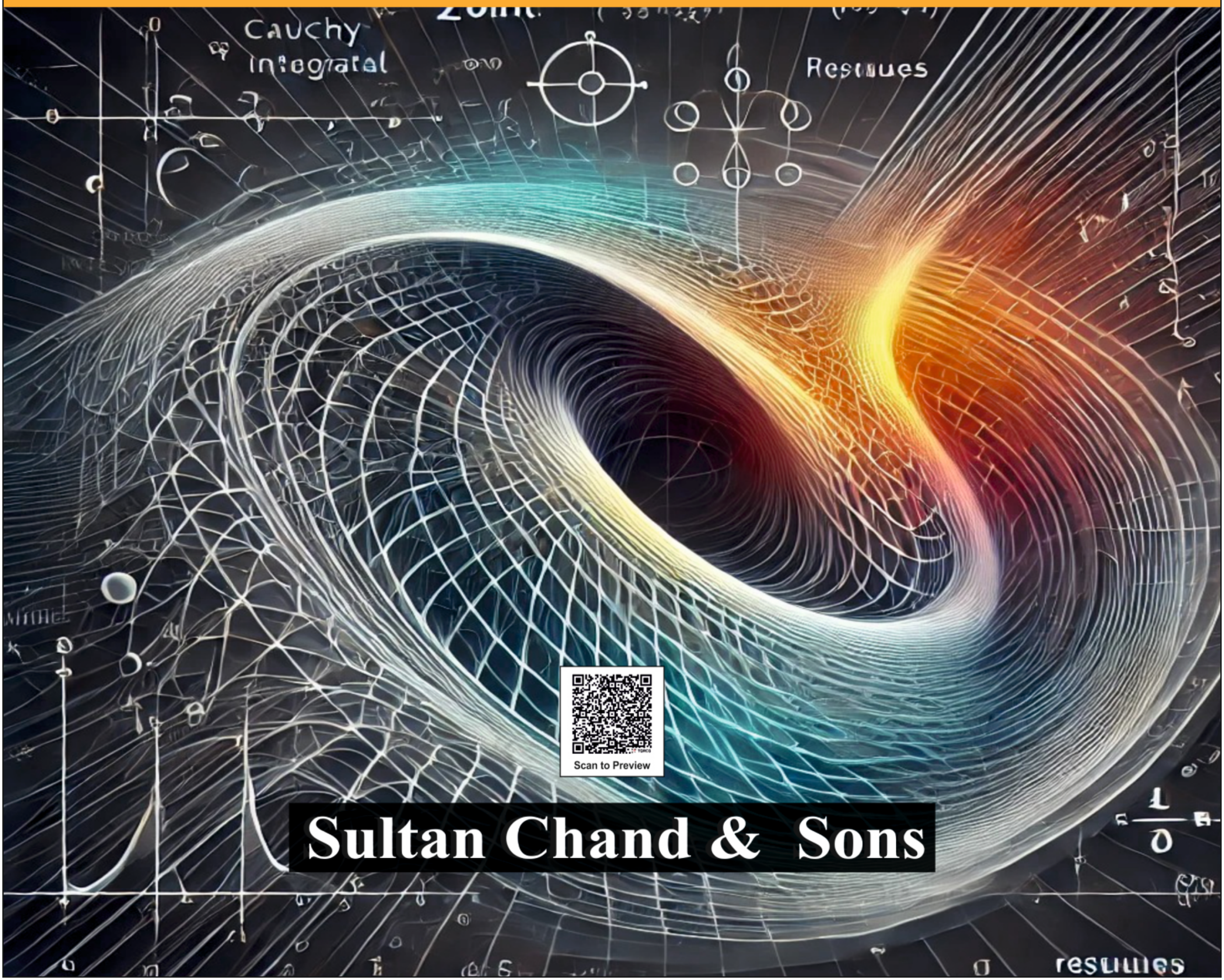
Essentials of

COMPLEX ANALYSIS

RITIKA NAGPAL • ARVIND YADAV



Scan to Preview

Sultan Chand & Sons

Essentials of Complex Analysis

♦ *To my loving Husband*

Ankit Gupta

(To whom I owe everything)

- Ritika Nagpal

♦ *To my respected Father*

Late Shri Babu Ram

(To whom I am forever indebted)

- Arvind Yadav

Essentials of Complex Analysis

Dr. Ritika Nagpal
Assistant Professor
Department of Mathematics
Vivekananda College
University of Delhi

Prof. (Dr.) Arvind Yadav
Professor
Department of Mathematics
Hansraj College
University of Delhi

SULTAN CHAND & SONS®
Educational Publishers
New Delhi

SULTAN CHAND & SONS®

Educational Publishers

23, Daryaganj, New Delhi-110 002

Phones : 011-23281876, 23266105, 41625022 *(Showroom & Shop)*
011-23247051, 40234454 *(Office)*

E-Mail : sultanchand74@yahoo.com; info@sultanchandandsons.com

Fax : 011-23266357; Website : www.sultanchandandsons.com

ISBN : 978-93-49290-81-5 (TC-1305)

Price : ₹ 495.00

First Edition: 2026

Author's Acknowledgement: The writing of a Textbook always involves creation of a huge debt towards innumerable authors and publications. We owe our gratitude to all of them. We acknowledge our indebtedness in extensive footnotes throughout the book. If, for any reason, any acknowledgement has been left out we beg to be excused. We assure to carry out correction in the subsequent edition, as and when it is known.

Printed at: Raja Offset Printers, Dilshad Garden Industrial Area Delhi-110095

Preface

Complex analysis is a fascinating and essential branch of mathematics, combining beauty with practical use. Its theories and applications are crucial in fields like physics, engineering, computer science, and economics. This book is written with care to make the subject clear and approachable, aiming to inspire students, teachers, and anyone eager to learn.

We have designed the book to match modern academic standards, following the Undergraduate Curriculum for B. Sc (Hons.) Mathematics and B. Tech (Engineering Mathematics). It is tailored to meet the needs of undergraduate students studying mathematics or related fields. It will also be helpful for those preparing for competitive exams or pursuing higher studies.

The book covers both fundamental ideas and advanced topics in a structured way. Starting with basic concepts like complex numbers, mappings, and transformations, it gradually moves into more advanced topics such as complex integration, series expansions, singularities, residues, and conformal mappings. Each chapter builds on the previous one, making the material easy to follow for readers with different levels of knowledge.

A unique aspect of this book is its focus on geometric intuition, using tools like the Argand plane, polar forms, and stereographic projections to help readers visualize concepts. Key ideas are explained with clear examples, diagrams, and proofs, making learning more engaging. Exercises at the end of each chapter, with hints and solutions, encourage readers to practice and test their understanding.

We have also highlighted practical applications of complex analysis to show its importance in solving real-world problems. Topics such as contour integration, series expansions, and conformal mappings are linked to real-life examples, helping readers connect theory to practice. To further enrich the learning experience, we have also featured Mathematica code to explore practical problems in complex analysis. This chapter aims to deepen conceptional understanding through visualization and computational experimentation.

It is our sincere hope that this book will make complex analysis interesting and accessible to all readers. We warmly welcome feedback, suggestions, and constructive criticism, which will help us improve future editions.

Let this book be your guide to the fascinating world of complex analysis. We hope you enjoy the journey!

Dr. Ritika Nagpal
Dr. Arvind Yadav

Acknowledgement

Writing this book has been a journey of exploration, learning, and collaboration. We express our deepest gratitude to the Almighty for divine guidance, strength, and inspiration throughout this endeavour.

We extend our heartfelt appreciation to our families, whose constant encouragement, patience, and understanding have been our greatest sources of strength. Their unwavering faith in us made this work possible.

We are deeply thankful to our students, whose curiosity and enthusiasm have inspired us to explain complex ideas in a more accessible and meaningful way. Their engagement has significantly shaped the content and direction of this book.

We sincerely acknowledge the invaluable guidance and support received from our mentors, colleagues, and academic peers. The encouragement and academic environment provided by Vivekananda College and Hansraj College, University of Delhi, have been instrumental in completing this work with dedication and focus.

We extend our special thanks to Prof. (Dr.) Rama, Principal, Hansraj College, University of Delhi; Prof. (Dr.) J.K. Singh, Department of Mathematics, Netaji Subhas University of Delhi, New Delhi; Prof. Pinki Maurya, Principal, Vivekananda College, University of Delhi; all faculty members of the Department of Mathematics, Vivekananda College, University of Delhi; Prof. Hina Nandrajog, Department of English, Vivekananda College, University of Delhi; and Prof. Shibesh Kumar Jas Pacif, Pacif Institute of Cosmology and Selfology, Odisha. Their suggestions have enriched the content and enhanced the clarity of the concepts discussed in this book.

Our sincere appreciation goes to Sultan Chand & Sons and their team for their professional assistance in bringing this book to its final form.

Finally, we acknowledge with respect and admiration the contributions of numerous mathematicians and researchers, whose pioneering work laid the foundation for this field and continues to inspire our understanding of complex analysis.

Dr. Ritika Nagpal
Dr. Arvind Yadav

Brief Contents

B

Syllabus

B.Sc. (Hons.) Mathematics,

B. Tech (Engineering Mathematics)

Unit 1: Foundations of Real and Complex Numbers

Core concepts in real numbers, limits, continuity, integrability, elementary functions, sequences in real analysis complex numbers, Argand plane, polar form, symmetry of complex numbers, Euler's formula, stereographic projection.

Unit 2: Complex Functions and their Behavior

Complex functions, mappings, parametric curves, limits, continuity, differentiability, analytic functions, Cauchy-Riemann equations, harmonic functions exponential functions, logarithmic functions, power functions, trigonometric functions, hyperbolic functions.

Unit 3: Integration, Series, and Conformal Mappings

Complex integration, contour integration, Cauchy-Goursat theorem, Cauchy's integral formulas; power series, Taylor series, Laurent series, singularities, residues, poles, zeros of analytic functions conformal mapping, bilinear transformations, Möbius transformations, and fixed points.

Mathematica Practicals

Lab Sessions are designed to be performed in Mathematica Software.

Contents

List of Figures

List of Practicals

P

This list of practicals highlights key computational and visualisation based activities essential for understanding concepts in Complex Analysis.

S. No.	*Title of the Practical*	*Concept / Objective*
1	Declaring Complex Numbers, Their Algebra and Plots	Perform basic operations on complex numbers and visualize them in the Argand plane.
2	Finding Conjugate, Modulus & Phase Angles	Compute conjugate, modulus, and argument; visualize them geometrically.
3	Compute Integral Over a Straight Line	Parametrize a line segment and evaluate contour integrals along it.
4	Plotting and Evaluating Contour Integrals for Line Segments	Evaluate integration along linear contours and visualize paths.
5	Contour Integration Over a Circle	Compute contour integrals along circular paths and verify Cauchy's theorem.
6	Contour Integration Over a Semicircle	Evaluate integrals on semicircular paths and apply the residue theorem.
7	Contour Integration Over Two Different Paths	Compare integral results over distinct contours and demonstrate path independence.
8	Geometric Visualization of nth Roots of Unity	Plot nth roots of unity as vertices of regular polygons on the unit circle.
9	Finding and Plotting Cube Roots of a Complex Number	Compute and visualize cube roots using De Moivre's theorem.
10	Parametric Equations and Plotting of an Ellipse with Transformations	Apply rotation and translation transformations to ellipses in the complex plane.
11	Visualizing the Mapping of a Disk Under a Linear Transformation	Examine scaling, rotation, and translation effects in linear mappings.

12	Mapping a Half-Plane to a Disk Using $w = \frac{1}{z}$	Explore geometric inversion and verify Möbius transformation mapping.
13	Using the ML Inequality to Bound a Contour Integral	Apply ML inequality to estimate bounds of contour integrals.
14	Mapping the Right Half-Plane to a Disk Using Möbius Transformation	Visualize Möbius transformations and their effect on geometric regions.
15	Finding and Plotting Three Different Laurent Series Expansions	Compute and visualize Laurent expansions around multiple points.
16	Finding Poles and Their Order	Identify poles, determine their order, and compute residues.
17	Finding Zeros, Poles, and Residues	Find zeros and poles, determine their order, and verify residue calculations.

List of Symbols

1 Mathematical Symbols

- $\Rightarrow$: implies
- $\Leftrightarrow$: is equivalent to
- {} : set
- $\in$: is an element of
- $\ni$: such that
- $\subset$: is contained in(is a subset of)
- $\supset$: contains(is a superset of)
- X − A : complement of *A* with respect to *X*
- $\cup$: union
- $\cap$: intersection
- $\varnothing$: the empty set
- $\exists$: there exists
- $\forall$: for all
- $\|\bullet\|$: norm
- ∗,+,−,/ : basic arithmetic operations
- <,=,> : comparison symbols
- ∞ : infinity
- ! : factorial
- $\therefore$: therefore
- $\because$: because
- $\cdot$: dot product or scalar multiplication

2 Greek Alphabets

- Alpha : Α, α
- Beta : Β, β
- Gamma : Γ, γ
- Delta : Δ, δ
- Epsilon : Ε, ϵ
- Eta : Η, η
- Theta : Θ, θ
- Iota : Ι, ι
- Lambda : Λ, λ
- Mu : Μ, μ
- Nu : Ν, ν
- Xi : Ξ, ξ
- Pi : Π, π
- Rho : Ρ, ρ
- Sigma : Σ, σ
- Tau : Τ, τ
- Phi : Φ, ϕ
- Chi : Χ, χ
- Psi : Ψ, ψ
- Omega : Ω, ω

3 Key notations on complex analysis

- $z = x+iy$: Complex number representation
- $\Re(z)$: The Real part of the complex number z
- $\Im(z)$: The Imaginary part of the complex number z
- $|z|$: Modulus of z
- $\overline{z}$: Complex conjugate of z
- Arg(z) : Argument (or phase) of z
- $e^{i\theta}$: Euler's formula
- C^* : Nonzero complex numbers

- $\mathbb{C}$: Complex numbers
- $\mathbb{R}$: Real numbers
- $\mathbb{N}$: Natural numbers
- $\mathbb{Z}$: Integers
- Domain Ω : A domain
- Region Ω : A region discussed
- Openset : O, U, ω
- Closed set : V, closure of A
- $N_\epsilon(u)$: ε-neighborhood of u
- $N_\in^*(z_0)$: ε-neighborhood of z_0 excluding z_0
- S^c : The complement of S
- *DOD* : Domain of Definition
- R : Connected Region
- Q : A rectangular path
- D : A disk
- $F(z)$: An antiderivative
- γ : A path (can be a closed, rectifiable curve in the complex plane $\mathbb{C}$)
- Winding number : Index(z_0;γ)
- Λ : The complement of the image of γ, *i.e.*, $\Lambda = \mathbb{C} \setminus \gamma(\mathbb{C})$
- $\log z = \ln|z| + i\arg(z)$: Complex logarithm
- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$: Exponential function n = 0n!
- $\sin z$, $\cos z$: Complex trigonometric functions
- $\sinh z$, $\cosh z$: Complex hyperbolic functions
- $f: \mathbb{C} \to \mathbb{C}$: Complex function
- $\lim_{z \to z_0} f(z) = f(z_0)$: Limit of $f(z)$
- $\frac{df}{dz}$: Complex derivative
- $f'(z)$: First derivative of f
- $f''(z)$: Second derivative of f
- $f^{(n)}$: nth derivative of f
- $\oint_\gamma f(z)\, dz$: Contour integral
- $Res(f, z_0)$: Residue off at z_0
- $\sum_{n=-\infty}^{\infty} c_n z^n$: Laurent series expansion
- $Ind(\gamma, z_0)$: Winding number
- $w = f(z)$: Complex function mapping
- $T(z) = \frac{az+b}{cz+d}$: M¨obius transformation
- $D = \{z \in \mathbb{C} : |z| < 1\}$: Open unit disk

Learning Objectives of the Book

L

Complex analysis is a cornerstone of modern mathematics, intertwining theory with deep applications in physics, engineering, and applied sciences. This book offers a rigorous yet intuitive exploration of the subject, equipping readers with both foundational knowledge and advanced problem-solving techniques. By the end of this book, readers will be able to:

1. Establish a Strong Mathematical Foundation
    - o Gain a solid understanding of prerequisites from real analysis, including limits, continuity, and integrability, to facilitate a smooth transition into complex analysis.
    - o Examine the historical evolution of complex numbers and their impact on mathematical advancements.
2. Master Complex Numbers and Their Representations
    - o Develop geometric intuition of complex numbers through representations on the Argand plane, polar form, and stereographic projections.
    - o Investigate key transformations and symmetries that govern complex numbers.
3. Understand Functions, Mappings, and Transformations
    - o Define and analyze complex functions, their domains, and their mapping properties in the complex plane.
    - o Learn how conformal mappings preserve angles and how bilinear transformations reshape geometric structures.
4. Explore the Depth of Analytic Functions
    - o Differentiate between holomorphic and analytic functions and their implications in complex function theory.
    - o Apply the Cauchy-Riemann equations to assess analyticity and study the harmonic nature of real and imaginary components.
5. Master Complex Integration Techniques
    - o Evaluate contour and line integrals using fundamental results such as Cauchy's Integral Theorem and Cauchy's Integral Formulas.
    - o Examine the consequences of Liouville's Theorem, the Maximum Modulus Principle, and the Fundamental Theorem of Algebra.
6. Unravel the Power of Series Representations
    - o Work with Power, Taylor, and Laurent series to analyze function behavior and convergence.
    - o Utilize series expansions to approximate functions, classify singularities, and develop complex function theory.

7. Classify and Investigate Singularities, Poles, and Residues
    - Identify and categorize isolated singularities and understand their effects on function behavior.
    - Apply the Residue Theorem to evaluate complex integrals and solve real-world problems efficiently.
8. Examine Conformal Mapping and Bilinear Transformations
    - Appreciate the geometric significance of conformal mappings in physics, engineering, and fluid dynamics.
    - Explore Möbius transformations, their special cases, and the concept of fixed points in complex transformations.
9. Apply Complex Analysis to Real-World Problems
    - Leverage the power of complex analysis to solve integrals, differential equations, and boundary value problems.
    - Investigate the practical applications of complex functions in electromagnetic theory, fluid mechanics, and signal processing.
10. Integrate Computational and Visual Tools in Complex Analysis
    - Utilize Mathematica to perform Computational experiments and Visualize Key concepts in Complex Analysis
    - Strengthen under standing of theoretical results through interactive lab work and practical problem-solving.

This book bridges the gap between theoretical depth and practical insight, making it an essential resource for students, researchers, and professionals alike. Through detailed proofs, illustrative examples, and challenging exercises, readers will develop a profound appreciation for the elegance and power of complex analysis.

About the Book

This book has been meticulously designed in alignment with the Undergraduate Curriculum ensuring it meets the academic requirements of students pursuing undergraduate mathematics and related disciplines. With a perfect balance of theoretical rigor and practical insights, this text serves as a self-sufficient guide for mastering the subject of Complex Analysis.

Keeping in mind the need to engage students, the book presents clear and lucid explanations of concepts, supported by detailed proofs and illustrative examples. It provides a seamless progression from fundamental topics like complex numbers, mappings, and transformations to advanced concepts such as complex integration, conformal mappings, series expansions, singularities, and residues.

A distinctive feature of this book is Chapter 9, "Bridging Theory and Practice: Computational and Visual Analysis," which introduces hands-on Mathematica-based practicals that enable learners to visualize, simulate, and verify complex analysis concepts through computational experimentation.

This book is especially designed to aid students of B.Sc. (Hons.) Mathematics, B.Tech., and other multidisciplinary courses while also serving as a valuable resource for competitive examinations. Whether you are a student, educator, or mathematics enthusiast, this book offers an engaging and comprehensive journey into the world of Complex Analysis, fostering both academic success and a deeper appreciation for the subject.

Salient Features

- ***Comprehensive Textbook:*** Covers theoretical and practical aspects of complex analysis.
- ***Lucid and Simple Language:*** Makes complex concepts easily understandable.
- ***Systematic Approach:*** Presents topics like analytic functions, integration, and mappings in a structured manner.
- ***Rich Problem Sets:*** Includes solved examples and exercises for practice and self-assessment.
- ***Application-Focused:*** Highlights real-world uses in physics, engineering, and applied sciences.
- ***Self-Study Friendly:*** Equipped with step-by-step explanations and key insights for independent learning.

About the Authors

Dr. Ritika Nagpal is an Assistant Professor in the Department of Mathematics at Vivekananda College, University of Delhi. She earned her Ph.D. from Netaji Subhas Institute of Technology, Faculty of Technology, University of Delhi. With over 4 years of teaching experience in Engineering Mathematics for B.Tech students and more than 6 years of experience teaching undergraduate courses [B.Sc.(H.) Mathematics], Dr. Nagpal has developed a strong foundation in Mathematical education. Her research expertise lies in the field of Cosmology, General theory of relativity, Observational Cosmology, where she is actively engaged in exploring cutting-edge topics. Her scholarly work has been published in reputed International Journals with high impact factors, reflecting her significant contributions to the field. Dr. Nagpal is dedicated to advancing mathematical understanding through her research and teaching, inspiring her students to appreciate the depth and beauty of Mathematics.

Dr. Arvind is a Professor in the Department of Mathematics at Hansraj College, University of Delhi. He has a distinguished career, including 5 years of experience as a Scientist at DRDO. With 20 years of teaching and research experience, Dr. Arvind has made significant contributions to the fields of Cryptography, Information Security, and Algebra. His research has been published in reputed National and International Journals, showcasing his expertise and commitment to advancing mathematical knowledge. He has authored two acclaimed books on Mathematics, namely Calculus and Numerical Methods, which are widely appreciated by students and educators alike. Dr. Arvind has successfully completed two research projects funded by the University of Delhi and continues to guide aspiring researchers as a mentor in the Department of Mathematics, University of Delhi.

Dedicated to the Service of Education • 1950 •
SULTAN

Sultan Chand & Sons
Publishers of Standard Educational Textbooks

23 Daryaganj, New Delhi-110002
Phones (S) : 011-23281876, 23266105, 41625022
(O) : 011-23247051, 40234454
Email : sultanchand74@yahoo.com
info@sultanchandandsons.com

Scan to Visit Us

TC 1305

ISBN 978-93-49290-81-5
9 789349 290815