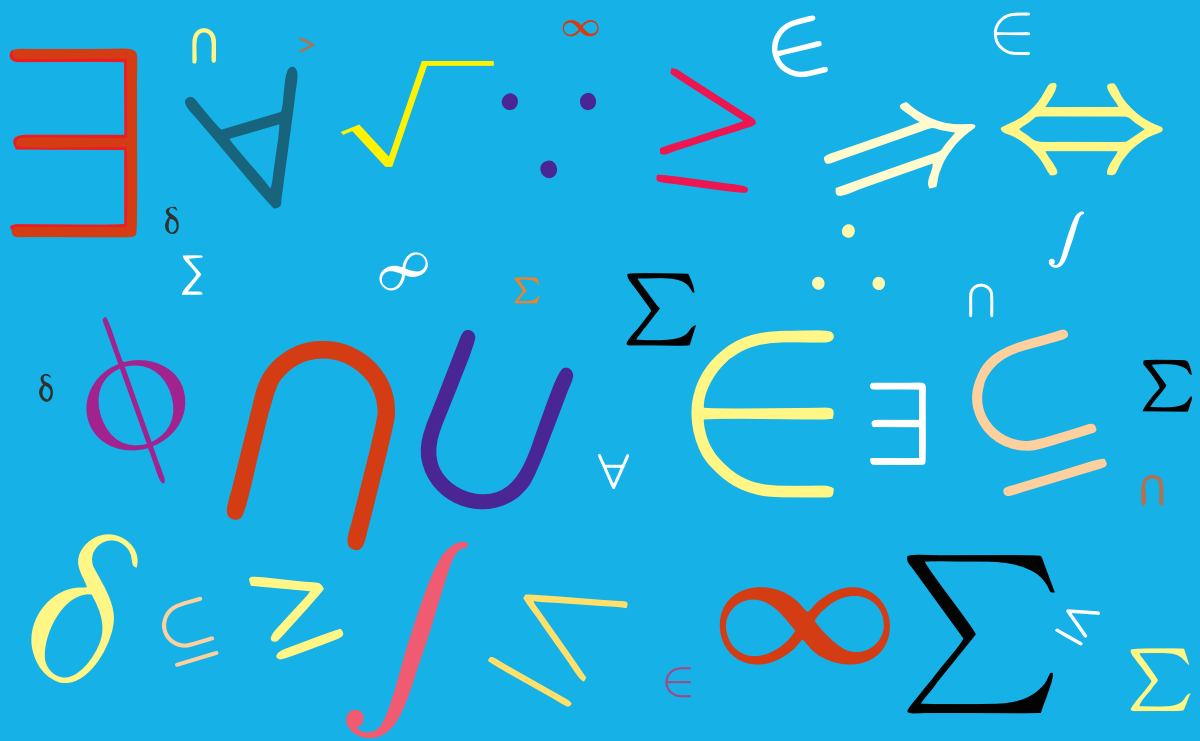
PN's

Elements of

REAL ANALYSIS

Pradip Narain

Typset in LATEX and Edited by
Amitabha Tripathi



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PN's

Elements of
REAL ANALYSIS

Dedicated to
the memory of
my beloved mother Chitra

To my dear grandson Sahir,

who I hope will read and
enjoy this book after about 10 years!

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Elements of

REAL ANALYSIS

(For B.Sc. / B.A. Program, B.Sc. Phy. Sc. & G.E. Courses of Delhi University)

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Sultan Chand & Sons®
Educational Publishers
New Delhi

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Educational Publishers
23, Daryaganj, New Delhi-110 002
Phones : 011-23281876, 23266105, 41625022 *(Showroom & Shop)*
011-23247051, 40234454 *(Office)*
E-mail : sultanchand74@yahoo.com; info@sultanchandandsons.com
Fax : 011-23266357; Website : www.sultanchandandsons.com

ISBN: 978-93-49290-44-0 (TC 1354)
***Price*: ₹ 195.00**
First Edition: 2025

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Printed at : Raja Offset, Dilshad Garden Industrial Area, Delhi-110095

Preface

Real Analysis, regarded as one of the core branches of higher Mathematics, is being taught at the undergraduate level of all Indian universities. While there are several text books available for the B.Sc. (Hons.) Mathematics level, there are hardly any which have been written exclusively for the Common Program Group, comprising of the B.Sc./ B.A. (Program), B.Sc. (Physical Sciences) and Generic Elective (G.E.) courses. This book has been written with the primary purpose of removing this lacuna. While it strictly follows the curriculum of the University of Delhi under the NEP 2020, it would be useful for students of other universities as well.

The book has been divided into 3 chapters covering the 3 topics in an introductory course of Real Analysis. The first chapter is an axiomatic study of Complete Ordered Fields, with the Real Number System R as a particular case. The second chapter deals with Real Sequences & their Convergence, followed by Monotone Sequences and Cauchy Sequences. The third chapter is devoted to Infinite Series which comprises mainly of Tests of Convergence of Positive term Series, Alternating Series and Absolute Convergence.

The book has kept in mind the needs of all students. Hence it has been written in a simple, step-by-step style with a focus on clarity.

Many thanks are due to the management and editorial team at Sultan Chand & Sons, particularly Shri Pratap Vaish and Dr. Shubhra Vaish, for their full cooperation as well as the excellent printing, layout and cover design of the book.

Pradip Narain
Amitabha Tripathi

To the Student

Dear students, this book has been written just for you (not for your teacher!).

Real Analysis (like Abstract Algebra) is very different from the traditional Mathematics you have studied so far. It is based more on logical reasoning than on computations. So please read the Appendix on Symbolic Logic at the outset. While going through the text, you must thoroughly understand and memorize the definitions. Also, after carefully reading the statement of each Theorem, DO NOT read the proof line by line. Rather, with pen and paper handy, keep writing what you read, doing as many steps as possible yourself.

Some of the material in this book may not be in your syllabus/guidelines, so check on your University website or ask your teacher. Also, the relatively challenging theorems/examples have been marked with *, with 'Note' written next to tricky steps.

Remember that proofs of Theorems are the heart and soul of Real Analysis and will constitute a substantial portion of the examination paper. So please practice the proofs properly.

A note about the symbol ϵ (epsilon): This Greek alphabet is used extensively in Real Analysis to denote an 'arbitrarily small' positive number. In this book, it is used in most of the proofs in Chapter 2 & Chapter 3, as well as in a few proofs in Chapter 1. With some practice, you will gradually get comfortable with its usage.

In each Section, the Theorems are followed by numerous Solved Examples. Some of these have been left incomplete (indicated by 'etc.') for you to complete yourself. Some Theorem proofs and examples have been left as Exercises.

In the examination, please present your proofs clearly and step-by-step. Justify the relevant steps with the name of the Theorem used (the theorem numbers have been provided for your reference). Also, you may use abbreviations (as in the text).

We are confident that you will enjoy learning Real Analysis from this book. Wishing you success in the upcoming examination.

Pradip Narain
Amitabha Tripathi

Acknowledgement

First and foremost, I am deeply indebted to my former student at St. Stephen's College, Prof. Amitabha Tripathi of IIT Delhi, for motivating me to write this book, taking out valuable time over several months to do a marvelous job of typesetting the entire manuscript in LATEX, and for doing the monumental task of editing the book to perfection.

I must express my gratitude to Prof. Prabir Bhattacharya, who made Real Analysis simple and interesting when I was his student at St. Stephen's College.

Many thanks are due to my wife Dr. Sunita Narain, Department of Mathematics, Jesus & Mary College, for her continuous encouragement and support, and for modifying my original notes in accordance with the latest syllabus under the NEP.

I would like to thank my sister (and first student!) Ms. Nandita Narain, formerly Head, Department of Mathematics, St. Stephen's College, for our invigorating discussions on Symbolic Logic & the teaching of Proofs.

Last but not least, I would like to thank Ms. Krishma Babbar, Department of Mathematics, St. Stephen's College, for painstakingly doing the proofreading to ensure that this book is error free.

Pradip Narain

Contents

List of Notations

Table 1: Standard Symbols

$\forall$	for all
$\exists$	there exists
$\Rightarrow$	implication, if . . . then
$\Leftrightarrow$	bi implication, if and only if
ϕ	empty set
$\subseteq$	subset
$\cup$	union
$\cap$	intersection

Table 2: Number Systems

$\mathbb{N}$	set of natural numbers = $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	set of integers = $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$\mathbb{Q}$	set of rational numbers = $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers

Table 3: Some Greek Alphabets

α	alpha
β	beta
γ	gamma
δ	delta
ϵ	epsilon
θ	theta
σ	sigma
Σ	Sigma (Capital case)

Table 4: List of Abbreviations

n.e.	non empty
s.t.	such that
iff	if and only if
Defn.	Definition
Lem.	Lemma
Thm.	Theorem
Cor.	Corollary
Obs.	Observation
Ill.	Illustration
soln.	solution
contra.	contradiction
add.	addition/additive
mult.	multiplication/multiple/multiplicative
Comm.	Commutative
Ass.	Associative
Distr.	Distributive
Prop.	Property
Inv.	Inverse
Pos.	Positive
Neg.	Negative
UB	upper bound
LB	lower bound
sup	supremum
inf	infimum
max	maximum
min	minimum
sc.	scalar
Arch.	Archimedean
Seq.	Sequence
Conv.	Convergent
Div.	Divergent
Mon.	Monotone
Lim.	Limit
cont.	continuous
diff.	differentiable
integ.	integrable
Comp.	Comparison
Alt.	Alternating
Abs.	Absolute
Cond.	Conditional

About the Book

PN's Elements of Real Analysis has been written strictly according to the syllabus of the Common Program Group, comprising of the B.Sc./B.A. (Program), B.Sc. (Physical Sciences) and Generic Elective (G.E.) courses of the University of Delhi (under the curriculum prescribed by the NEP 2020). It would also be very useful as an introductory text for students of B.Sc. (Hons.) Mathematics 1st Sem. of Delhi University, as well as other Indian Universities.

The book covers the three basic topics in an introductory course of Real Analysis:

1. Structure of Real Numbers as a Complete Ordered Field.
2. Real Sequences.
3. Infinite Series.

Salient Features

- Contains close to 40 Definitions, 80 Theorems and 80 Solved Examples (most of which have several parts), given in a graded, step by step format.
- Several exercises at the end of each Section provided for practice and supplementing the theory.
- Perfect for self study; no teacher required.
- Covers all the Theorems and Guideline problems in detail in very few pages.
- Almost all questions of the Common Program Group of Delhi University since 2020 included and solved at relevant places.

About the Author

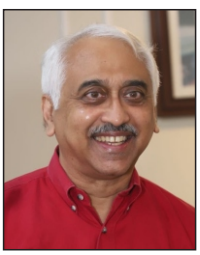
Mr Pradip Narain, popularly known as PN Sir, taught undergraduate and postgraduate students of Mathematics for over forty years. After topping Delhi University in MA Mathematics from St Stephen's College, he taught in the Departments of Mathematics, Economics & Commerce at Hindu College, St Stephen's College and Jesus & Mary College, as well as at the Department of Business Economics, University of Delhi.

Mr Narain also holds an MPhil in Mathematics Education from the prestigious University of Cambridge (UK), where he went on a British Council Fellowship. His thesis on 'Teaching of Proofs in Real Analysis' won the best thesis award. A paper based on this was published by The Korea Society of Educational Studies in Mathematics.

After taking voluntary retirement from Jesus & Mary College, Mr Narain became the founder-director of the Institute of Mathematics & Economics, New Delhi, which provided quality coaching to undergraduate students of Delhi University.

PN Sir is widely regarded as one of the finest Mathematics teachers, who transformed the lives of thousands of students during his long and illustrious career. His crystal clear lecture notes in diverse areas of Mathematics were the lifeline for generations of students of Delhi University. This book is based on his lecture notes in Elementary Real Analysis, modified according to the current curriculum prescribed by the NEP 2020.

About the Editor



Amitabha Tripathi holds the position of Vaidya Saolapurkar Chair Professor in the Department of Mathematics at Indian Institute of Technology Delhi. He studied Mathematics at St. Stephen's College, University of Delhi, IIT Kanpur & State University of New York at Buffalo, from where he obtained a Ph.D. in Mathematics. He traces his mathematical roots to the celebrated English mathematician J.E. Littlewood. His research interests lie in elementary and combinatorial aspects of Number Theory, in combinatorial aspects of Graph Theory, and more recently, in Ramsey Theory and in Numerical Semigroups. Prof. Tripathi has been with the Department of Mathematics at IIT Delhi since July 1991, where in addition to teaching a wide variety of courses, he has mentored and helped guide research among undergraduate students that has led to several fruitful collaborations.

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ISBN 978-93-49290-44-0